The Boundary State Formalism and Conformal Invariance in Off-shell String Theory

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Abstract

In this note we present a generalization of the boundary state formalism for the bosonic string that allows us to calculate the overlap of the boundary state with arbitrary closed string states. We show that this generalization exactly reproduces world-sheet sigma model calculations, thus giving the correct overlap with both on- and off-shell string states, and that this new boundary state automatically satisfies the requirement for integrated vertex operators in the case of non-conformally invariant boundary interactions

1 Introduction

The problem of studying off-shell string theory is an old one, and there have been many attempts to examine it. A particularly interesting approach is background independent string field theory [?, ?, ?, ?, ?] which has received attention recently as an approach which facilitates the understanding of properties of unstable d-branes [?, ?, ?]. A tractable problem that can be approached within this formalism is to understand the behavior of the off-shell theory in the background of a tachyon field, in particular a quadratic function of the coordinates. This model has been the subject of some research interest [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?] and the resulting theory is not invariant under conformal transformations except in the trivial cases of vanishing or infinite quadratic tachyon potential. This subtlety reveals an interesting structure. In this note we suggest a generalization of the boundary state [?, ?, ?, ?, ?, ?] which naturally accommodates this loss of invariance. This new boundary state can be applied to the problem of computing off-shell closed string emission amplitudes from a d-brane.

For definiteness, we consider the following action on the string world sheet

$$\begin{aligned} S(g,F,T_0,U) &= \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma g_{\mu\nu}\partial^a X^\mu \partial_a X^\nu \\ &+ \int_{\partial\Sigma} d\theta \left(\frac{1}{2}F_{\mu\nu}X^\nu\partial_t X^\mu + \frac{1}{2\pi}T_0 + \frac{1}{8\pi}U_{\mu\nu}X^\mu X^\nu\right). \end{aligned} \tag{1}$$

Here, α' is the inverse string tension, Σ is the string world sheet, $d^2\sigma$ is the measure on the bulk, $d\theta$ is the measure on the boundary, and ∂_t is the tangential derivative to the boundary. The background fields that are included in this are $F_{\mu\nu}$, a constant gauge field strength, and $T(X) = \frac{1}{2\pi}T_0 + \frac{1}{8\pi}U_{\mu\nu}X^\mu X^\nu$, the tachyon profile parameterized by a constant and a symmetric matrix.

We wish to show that the boundary state that we propose will provide an algebraic method to calculate results that could be obtained in the σ-model calculations, and so we will be comparing the results of σ-model calculation in these backgrounds with analogous results obtained through boundary state calculations. This will fix the normalization of the boundary state and verify that it gives the results expected in the background of the tachyon condensate.

2 Sigma Model calculations

It is instructive to commence by calculating some closed string emission amplitudes from the disk world-sheet, since the boundary state will be seen to interact with closed string modes in this way. First, it is useful to fix some conventions, the functional integral which we compute is in all cases an average over the action given in (1),

$$\langle \mathcal{O}(X) \rangle = \int \mathcal{D}X e^{-S(X)} \mathcal{O}(X). \tag{2}$$

In addition, the greens function on the unit disk with Neumann boundary conditions is determined to be [**?**]

$$G^{\mu\nu}(z, z') = -\alpha' g^{\mu\nu} \left(-\ln |z - z'| + \ln |1 - z\bar{z}'| \right), \tag{3}$$

and it will be useful also to know the bulk to boundary propagator which is

$$G^{\mu\nu}(\rho e^{i\phi}, e^{i\phi'}) = 2\alpha' g^{\mu\nu} \sum_{m=1}^{\infty} \frac{\rho^m}{m} \cos[m(\phi - \phi')]. \tag{4}$$

The boundary to boundary propagator can be read off from (4) as the limit in which $\rho \to 1$. Throughout, we will use $z = \rho e^{i\phi}$ as a parameterization of the points within the unit disk, so $0 \le \rho \le 1$ and $0 \le \phi < 2\pi$. Using the bulk to boundary propagator it is possible to integrate out the quadratic interactions on the boundary [**?**] and to obtain an exact propagator, which is given by

$$\begin{aligned} G^{\mu\nu}(z, z') &= \alpha' g^{\mu\nu} \ln |z - z'| - \alpha' g^{\mu\nu} \ln |1 - z\bar{z}'| \\ &\quad -\alpha' \sum_{n=1}^{\infty} \left(\frac{2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}} \right)^{\mu\nu} \frac{(z\bar{z}')^n + (\bar{z}z')^n}{n} \\ &= \alpha' g^{\mu\nu} \ln |z - z'| + \frac{\alpha'}{2} \sum_{n=1}^{\infty} \left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}} \right)^{\mu\nu} \frac{(z\bar{z}')^n + (\bar{z}z')^n}{n}. \end{aligned} \tag{5}$$

The first interesting calculation that can be done is the partition function, which has been calculated in several ways in the literature. In the σ-model approach the oscillator modes of X must be integrated out with the contributions from F and U treated as perturbations. Since both perturbations are quadratic, all the feynmann graphs that contribute to the free energy can be written and evaluated, and explicitly (using the parameterization $z = \rho e^{i\phi}$) the free energy is given by the sum

$$\begin{aligned} \mathcal{F} &= \sum_{n=1}^{\infty} \frac{1}{n} \int d\phi_1 \dots d\phi_n (-1)^n \Bigg[\left(F_{\mu_1\nu_1} \partial_{\phi_1} + \frac{1}{4\pi} U_{\mu_1\nu_1} \right) \times \\ &\quad 2g^{\nu_1\mu_2} \sum_{m_1=1}^{\infty} \frac{\cos[m_1(\phi_1 - \phi_2)]}{m_1} \dots \left(F_{\mu_n\nu_n} \partial_{\phi_n} + \frac{1}{4\pi} U_{\mu_n\nu_n} \right) 2g^{\nu_n\mu_1} \times \\ &\quad \sum_{m_n=1}^{\infty} \frac{\cos[m_n(\phi_n - \phi_1)]}{m_n} \Bigg] \\ &= -\sum_{m=1}^{\infty} Tr \ln \left(g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{m} \right), \end{aligned} \tag{6}$$

see [?, ?] for further calculations done in this spirit. From (6) we immediately see that the partition function is given by

$$\begin{aligned} Z &= e^{-T_0} \prod_{m=1}^{\infty} \frac{1}{\det\left(g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{m}\right)} \int dx_0 e^{-\frac{U_{\mu\nu}}{4} x_0^{\mu} x_0^{\nu}} \\ &= \frac{1}{\det\left(\frac{U}{2}\right)} e^{-T_0} \prod_{m=1}^{\infty} \frac{1}{\det\left(g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{m}\right)}. \end{aligned} \tag{7}$$

This expression is divergent, but using ζ-function regularization [?] it can be reduced to

$$Z = e^{-T_0} \sqrt{\det\left(\frac{g + 2\pi\alpha' F}{U/2}\right)} \det\Gamma\left(1 + \frac{\alpha' U/2}{g + 2\pi\alpha' F}\right), \tag{8}$$

where $\Gamma(g)$ is the Γ function and the dependence of all transcendental functions on the matrices U and F is defined by their Taylor expansion.

We now wish to calculate the expectation value for vertex operators that correspond to different closed string states, however this is a process that must be done with some care. To calculate the emission of a closed string in the world-sheet picture one generally considers a disk emitting an asymptotic closed string state. This is really a closed string cylinder diagram. The standard method is to use conformal invariance to map the closed string state to a point on the disk, namely the origin, where a corresponding vertex operator is inserted. On the other hand it has been cogently argued that it is necessary to have an integrated vertex operator for closed string states to properly couple [?], in particular that the graviton must be produced by an integrated vertex operator to couple correctly to the energy momentum tensor. The distinction between a fixed vertex operator and an integrated vertex operator is moot in the conformally invariant case where the integration will only produce a trivial volume factor, however in the case we consider more care must be taken. We wish to consider arbitrary locations of the vertex operators on the string world sheet, and the natural measure to impose is that of the conformal transformations which map the origin to a point within the unit disk on the complex plane.

In other words we propose to allow the vertex operator corresponding to the closed string state to be moved from the origin by a conformal transformation that preserves the area of the unit disk, namely a PSL(2,R) transformation. The method to accomplish this is to go to a new coordinate system

$$y = \frac{az + b}{b^* z + a^*}, \quad |a^2| - |b^2| = 1, \tag{9}$$

and a vertex operator at the origin $y = 0$ would correspond to an insertion of a vertex operator at the point $z = \frac{-b}{a}$. It is worth noting that in the case of conformal invariance, that is when $U \to 0$ or $U \to \infty$ the greens function remains unchanged in form, the y dependence coming from the replacement $z \to z(y)$. Even in the case of finite U the only change to the greens function is the addition of a term that is harmonic within the unit disk. The parameter of the integration over the position of the vertex operator would be to the measure on PSL(2,R), giving an infinite factor in the conformally invariant case [?, ?, ?]. From this argument we have a definite prescription for the calculation of vertex operator expectation values, which is to use the conformal transformation to modify the greens function, and calculate the expectation values of operators at the origin with this modified greens function.

Now we will use this prescription to calculate the sigma model expectation values of some operators, and we will start with the simplest, that of the closed string tachyon. The vertex operator for the tachyon is $: e^{ip_\mu X^\mu(z(y))} :$, and it is inserted at the point $y = 0$. The normal ordering prescription for all such operators is that any divergent pieces will be subtracted, but finite pieces will remain and by inspection we see that the appropriately subtracted greens function is

$$\begin{aligned} :\mathcal{G}^{\mu\nu}(z,z'): &= G^{\mu\nu}(z,z') - g^{\mu\nu}\alpha' \ln|z-z'| \\ &= \frac{\alpha'}{2}\sum_{n=1}^{\infty}\left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}}\right)^{\mu\nu}\frac{(z\bar{z}')^n + (\bar{z}z')^n}{n}. \end{aligned} \tag{10}$$

Using (10) we can immediately see that

$$\begin{aligned} \langle : e^{ip_\mu X^\mu(y=0)} : \rangle &= Z e^{-\frac{1}{2}p_\mu p_\nu :\mathcal{G}^{\mu\nu}(z(y),z'(y)):}\Big|_{y=0} \\ &= Z\exp\left(-\frac{\alpha'}{2}p_\mu p_\nu \sum_{n=1}^{\infty}\left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}}\right)^{\mu\nu}\frac{1}{n}\frac{|b^{2n}|}{|a^{2n}|}\right). \end{aligned} \tag{11}$$

Thus we find that the expectation value for the tachyon vertex operator is

$$\begin{aligned} &\int d^2a d^2b \delta(|a^2| - |b^2| - 1) \\ &\quad \times \langle : e^{ip_\mu X^\mu(y(a,b)=0)} : \rangle = \int d^2a d^2b \delta(|a^2| - |b^2| - 1) Z \\ &\qquad \exp\left(-\frac{\alpha'}{2}p_\mu p_\nu \sum_{n=1}^{\infty}\left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}}\right)^{\mu\nu}\frac{1}{n}\frac{|b^{2n}|}{|a^{2n}|}\right). \end{aligned} \tag{12}$$

A similar analysis can also be performed for the massless closed string excitations. In particular the graviton insertion at $y = 0$ is given by

$$\langle \mathcal{V}_h \rangle = \langle : -\frac{2}{\alpha'} h_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu e^{ip_\mu X^\mu(y=0)} : \rangle \tag{13}$$

where h is a symmetric traceless tensor and the normalization follows the conventions of [?]. This can be analyzed by the same techniques as for the tachyon, noting that there will be cross contractions between the exponential and the X-field prefactors. Explicitly we obtain

$$\begin{aligned}
\langle \mathcal{V}_h \rangle &= -\frac{2}{\alpha'} Z h_{\mu\nu} \left(\partial\bar{\partial}' : \mathcal{G}^{\mu\nu}\left(z(y), z'(y)\right) : + \partial : \mathcal{G}^{\mu\alpha}\left(z(y), z'(y)\right) : \right. \\
&\quad \left. \times \bar{\partial} : \mathcal{G}^{\mu\beta}\left(z(y), z'(y)\right) : (ip_\alpha)(ip_\beta) \right) e^{-\frac{1}{2} p_\mu p_\nu : \mathcal{G}^{\mu\nu}(z(y), z'(y)) :}|_{y=0} \\
&= Z h_{\mu\nu} \left(- \sum_{n=1}^{\infty} \left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}} \right)^{\mu\nu} n \frac{|b^{2(n-1)}|}{|a^{2(n-1)}|} \frac{1}{|a^2|^2} \right. \\
&\quad + \frac{\alpha'}{2} \sum_{n=1}^{\infty} \left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}} \right)^{\mu\alpha} \frac{|b^{2(n-1)}|}{|a^{2(n-1)}|} \frac{-b}{|a^2| a} \\
&\quad \left. \times \sum_{m=1}^{\infty} \left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{m}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{m}} \right)^{\nu\beta} \frac{|b^{2(m-1)}|}{|a^{2(m-1)}|} \frac{-b*}{|a^2| a^*} p_\alpha p_\beta \right) \\
&\quad \exp\left(-\frac{\alpha'}{2} p_\mu p_\nu \sum_{n=1}^{\infty} \left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}} \right)^{\mu\nu} \frac{1}{n} \frac{|b^{2n}|}{|a^{2n}|} \right).
\end{aligned} \tag{14}$$

Clearly a similar analysis can be performed for either the Kalb-Ramond field or the dilaton, and the only change would be to replace $h_{\mu\nu}$ with the appropriate polarization tensor for either field.

Finally, we can perform the same kind of calculation for a more general closed string state, and while the analysis below is not performed for a completely general state, it contains the germs of generality. We consider a state which may be off shell in the sense that it not annihilated by the positive modes of the σ-model energy momentum tensor (the Virasoro generators), may not satisfy the mass shell condition, and may not be level matched. Our explicit choice is to consider the operator

$$\langle \mathcal{V}_A \rangle = \langle : -i \left(\frac{2}{\alpha'} \right)^{3/2} A_{\mu\nu\delta} \frac{\partial^a}{(a-1)!} X^\mu \frac{\bar{\partial}^b}{(b-1)!} X^\nu \frac{\bar{\partial}^c}{(c-1)!} X^\gamma e^{ip_\mu X^\mu} : \rangle \tag{15}$$

which is an arbitrary state involving three creation operators. We find that

$$\begin{aligned}
\langle \mathcal{V}_A \rangle &= Z A_{\mu\nu\delta} \left(\frac{2}{\alpha'} \right)^{3/2} \left(\frac{\partial^a}{(a-1)!} \frac{\bar{\partial}'^b}{(b-1)!} : G^{\mu\nu}(z, z') : \frac{\bar{\partial}^c}{(c-1)!} : G^{\delta\alpha}(z, z') : p_\alpha \right. \\
&\quad + \frac{\partial^a}{(a-1)!} \frac{\bar{\partial}'^c}{(c-1)!} : G^{\mu\delta}(z, z') : \frac{\bar{\partial}^b}{(b-1)!} : G^{\nu\alpha}(z, z') : p_\alpha \\
&\quad \left. - \frac{\partial^a}{(a-1)!} : G^{\mu\alpha}(z, z') : \frac{\bar{\partial}^b}{(b-1)!} : G^{\nu\beta}(z, z') : \frac{\bar{\partial}^c}{(c-1)!} : G^{\delta\gamma}(z, z') : p_\alpha p_\beta p_\gamma \right)
\end{aligned}$$

$$\times e^{-\frac{1}{2}p_\mu p_\nu :\mathcal{G}^{\mu\nu}(z,z'):}\Big|_{y=0} . \tag{16}$$

It is straightforward but not very instructive to take the derivatives acting on the greens functions and evaluate the result at $y = 0$. This general state will allow us to check the prescription for the boundary state presented in the next section.

3 Boundary States

We now perform the same kind of analysis from the point of view of the boundary state formalism. To this end, it is important to understand where the boundary state comes from. By varying the action (1) one obtains the equation

$$\frac{1}{2\pi\alpha'} g_{\mu\nu}\partial_n X^\nu + F_{\mu\nu}\partial_t X^\nu + \frac{1}{4\pi}U_{\mu\nu}X^\nu \quad = \quad 0 \tag{17}$$

on the boundary of the string world-sheet. As discussed in, for example, [**?**, **?**] the boundary state $|B\rangle$ is a state of closed string theory which obeys the boundary condition (17) as an operator equation. Using the following standard mode expansion for X as a function of z

$$X^\mu(z,\bar{z}) = x^\mu + p^\mu \ln|z^2| + \sum_{m\neq 0}\frac{1}{m}\left(\frac{\alpha_m^\mu}{z^m} + \frac{\tilde{\alpha}_m^\mu}{\bar{z}^m}\right). \tag{18}$$

we obtain as a boundary condition on the modes that

$$\left(g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}\right)_{\mu\nu} \alpha_n^\mu + \left(g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}\right)_{\mu\nu} \tilde{\alpha}_{-n}^\mu = 0. \tag{19}$$

We wish to make a state that obeys (19) as an operator equation, so that operating with a state of positive index gives the appropriate negative index coefficient to make

$$\left[\left(g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}\right)_{\mu\nu} \alpha_n^\mu + \left(g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}\right)_{\mu\nu} \tilde{\alpha}_{-n}^\mu\right]|B\rangle = 0, \tag{20}$$

$$\left[g_{\mu\nu}p^\mu - i\frac{\alpha'}{2}U_{\mu\nu}x^\mu\right]|B\rangle = 0. \tag{21}$$

It is relatively easy to see that the state to satisfy this must be a coherent state and is given by

$$|B\rangle = \mathcal{N}\prod_{n\geq 1}\exp\left(-\left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}}\right)_{\mu\nu}\frac{\alpha_{-n}^\mu\tilde{\alpha}_{-n}^\nu}{n}\right)\exp\left(-\frac{\alpha'}{4}x^\mu U_{\mu\nu}x^\nu\right)|0\rangle \tag{22}$$

where $\mathcal{N}$ is an as yet undetermined normalization constant.

It is interesting to examine how this boundary state transforms under the action of the residual conformal symmetry of the disk, namely under PSL(2,R) transformations. In the two conformally invariant cases ($U = 0$ and $U = \infty$) this is a good symmetry of the action, but we expect there to be interpolation as the flow from $U = 0$ to $U = \infty$ takes us from Dirichlet to Neumann boundary conditions. As mentioned previously the action of PSL(2,R) on the complex coordinates of the disk is to perform the mapping

$$z \rightarrow w(z) = \frac{az + b}{b^* z + a^*} \tag{23}$$

where a and b satisfy the relation

$$|a^2| - |b^2| = 1. \tag{24}$$

This transformation maps the interior of the unit disk to itself, the exterior to the exterior and the boundary to the boundary. Moreover, this transformation of the coordinates induces a mapping which intermixes the oscillator modes. To see this consider the definition of the oscillator modes

$$\alpha_m^\mu = \sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi} z^m \partial X^\mu(z) \tag{25}$$

where the contour is the boundary of the unit disk, and the mode expansion of X is

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_m \frac{\alpha_m^\mu}{z^{m+1}}. \tag{26}$$

Now, using the fact that X is a scalar, or equivalently the fact that ∂X is a (1,0) tensor, we see that

$$\alpha_m^\mu = \oint \frac{dz}{2\pi i} z^m \partial_w X^\mu(w) \frac{dw}{dz}. \tag{27}$$

Now, using the fact that a mode expansion for X exists in terms of w with coefficients α'_m in exactly the same way as (26), we see that

$$\alpha_m^\mu = M_{mn} \alpha_n^{\prime\mu} \tag{28}$$

where

$$M_{mn} = \oint \frac{dz}{2\pi i} z^m \frac{(b^* z + a^*)^{n-1}}{(az + b)^{n+1}}. \tag{29}$$

Upon evaluation it becomes clear that this matrix has a block diagonal form, which is due to the fact that there are no poles inside (outside) the integration region when $n < 0 < m$ ($m < 0 < n$). The consequence of this is that under PSL(2,R) transformations the creation operators transform into creation operators and the annihilation operators likewise transform into annihilation operators. Furthermore, upon rescaling $\mathcal{M}_{mp} \to \sqrt{\frac{p}{m}} M_{mp}$ so that the oscillators are normalized as creation and annihilation operators it is easy to show that for both positive and negative p and m

$$\mathcal{M}^{-1}_{mp} = \oint \sqrt{\frac{m}{p}} \frac{d\bar{z}}{-2\pi i} \bar{z}^p \frac{(a + b\bar{z})^{m-1}}{(b^* + a^*\bar{z})^{m+1}} = \mathcal{M}^{*T}_{mp} = \mathcal{M}^\dagger_{mp}. \tag{30}$$

This is simply a statement of the fact that M preserves the inner product on the space of operators. It would be inappropriate for M itself to be hermitian because the commutation relation between the modes is

$$[\alpha^\mu_n, \alpha^\nu_m] = n\delta_{n-m} g^{\mu\nu}. \tag{31}$$

The general expression for M when both indices are positive can easily be found by explicit contour integration and is

$$M_{mp} = \sqrt{\frac{p}{m}} \sum_{k=0}^{p} \begin{pmatrix} m \\ k \end{pmatrix} \begin{pmatrix} p-1 \\ p-k \end{pmatrix} (-1)^{m-k} b^m |b^2|^{-k} a^{-p}, \tag{32}$$

where the binomial coefficients $\begin{pmatrix} a \\ b \end{pmatrix}$ are understood to vanish in all cases where $b > a$. It will be important to note that there is a non-zero overlap with the the zero mode which will be important when we make a correspondence between the sigma model calculation and the boundary state.

To accommodate the coordinate transformation under PSL(2,R) the boundary state becomes

$$\begin{aligned} |B_{a,b}\rangle \quad &= \quad \mathcal{N} \exp \left(\sum_{n=1,j,k=-\infty}^{\infty} \alpha^\mu_{-k} M_{-n-k}(a,b) \left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha}{2}\frac{U}{n}} \right)_{\mu\nu} \right. \\ &\left. \frac{1}{n} M^*_{-n-j}(a,b) \tilde{\alpha}'^\nu_{-j} \right) \exp\left(-\frac{\alpha'}{4} x^\mu U_{\mu\nu} x^\nu \right) |0\rangle \end{aligned} \tag{33}$$

and in this equation and all following ones we drop the ' associated with the transformation for notational simplicity. We conjecture that the proper definition of the boundary state to give the correct overlap with all closed string states is

$$|B\rangle = \int d^2a d^2b \delta(|a^2| - |b^2| - 1) |B_{a,b}\rangle. \tag{34}$$

This is just the boundary state (33) integrated over the Haar measure of PSL(2,R). We will now verify that this gives the correct overlap with the tachyon and massless states by comparing with the σ-model calculations of the previous section.

The computation for the tachyon is easy. To fix the normalization it suffices to take the inner product

$$
\begin{aligned}
\langle 0|B\rangle &= \mathcal{N}\int d^2a d^2b\delta(|a^2|-|b^2|-1)\exp\left(-\frac{\alpha'}{2}x^\mu U_{\mu\nu}x^\nu\right) \\
&\quad \exp\left(-\sum_{n=1}^{\infty}\frac{\alpha'}{2}p_\mu M_{-n0}\frac{1}{n}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{n}}\right)_{\mu\nu}M^*_{-n0}p_\nu\right) \qquad (35)
\end{aligned}
$$

and we have used the relations $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$ and $[x^\mu, p^\nu] = ig^{\mu\nu}$. We can insert the explicit form $M_{-n0} = \left(\frac{-b^*}{a^*}\right)^n$ to (35) and find

$$
\begin{aligned}
\langle 0|B\rangle &= \mathcal{N}\int d^2a d^2b\delta(|a^2|-|b^2|-1)\exp\left(-\frac{\alpha'}{2}x^\mu U_{\mu\nu}x^\nu\right) \\
&\quad \exp\left(-\sum_{n=1}^{\infty}\frac{\alpha'}{2}p_\mu\frac{1}{n}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{n}}\right)_{\mu\nu}p_\nu\frac{|b^{2n}|}{|a^{2n}|}\right). \qquad (36)
\end{aligned}
$$

By comparing this to (12) we can unambiguously fix the normalization as

$$
\mathcal{N} = Z. \qquad (37)
$$

We now perform an analogous calculation for the massless states which will provide a non-trivial check of this normalization scheme. For an arbitrary massless state with polarization tensor $P_{\mu\nu}$

$$
|P_{\mu\nu}\rangle = P_{\mu\nu}\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|0\rangle \qquad (38)
$$

the overlap to be calculated is

$$
\begin{aligned}
\langle P_{\mu\nu}|B\rangle &= \mathcal{N}\int d^2a d^2b\delta(|a^2|-|b^2|-1)\exp\left(-\frac{\alpha'}{2}x^\mu U_{\mu\nu}x^\nu\right) \\
&\quad \exp\left(-\sum_{n=1}^{\infty}\frac{\alpha'}{2}p_\mu M_{-n0}\frac{1}{n}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{n}}\right)_{\mu\nu}M^*_{-n0}p_\nu\right) \\
&\quad P_{\mu\nu}\left[-\sum_{n=1}^{\infty}M_{-n-1}\frac{1}{n}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{n}}\right)^{\mu\nu}M^*_{-n-1}\right. \\
&\quad +\frac{\alpha'}{2}\sum_{n=1}^{\infty}M_{-n-1}\frac{1}{n}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{n}}\right)^{\mu\alpha}M^*_{-n0}p_\alpha \\
&\quad \left.\times\sum_{m=1}^{\infty}M_{-m0}\frac{1}{m}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{m}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{m}}\right)^{\beta\nu}M^*_{-n-1}p_\beta\right]
\end{aligned}
$$

$$
\begin{aligned}
= \;& \mathcal{N}\int d^2a d^2b\delta(|a^2|-|b^2|-1)\exp\left(-\frac{\alpha'}{2}x^\mu U_{\mu\nu}x^\nu\right) \\
& \exp\left(-\sum_{n=1}^{\infty}\frac{\alpha'}{2}p_\mu\frac{1}{n}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{n}}\right)_{\mu\nu}p_\nu\frac{|b^{2n}|}{|a^{2n}|}\right) \\
& P_{\mu\nu}\left[-\sum_{n=1}^{\infty}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{n}}\right)^{\mu\nu} n\frac{|b^{2(n-1)}|}{|a^{2(n-1)}|}\frac{1}{|a^2|^2}\right. \\
& +\frac{\alpha'}{2}\sum_{n=1}^{\infty}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{n}}\right)^{\mu\alpha}\frac{|b^{2(n-1)}|}{|a^{2(n-1)}|}\frac{-b^*}{|a^2|a^*}p_\alpha \\
& \left.\times\sum_{m=1}^{\infty}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{m}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{m}}\right)^{\beta\nu}\frac{|b^{2(m-1)}|}{|a^{2(m-1)}|}\frac{-b}{|a^2|a}p_\beta\right] \qquad (39)
\end{aligned}
$$

Now, we compare (39) with (14) to again find that the normalization is fixed by

$$\mathcal{N} = Z.$$

We can also use the boundary state to compute the overlap with the more general state that we considered in the sigma model calculations. Explicitly the overlap between the boundary state and state A defined by

$$|A_{\mu\nu\delta}\rangle \;=\; A_{\mu\nu\delta}\alpha^\mu_{-a}\tilde{\alpha}^\nu_{-b}\tilde{\alpha}^\delta_{-c}|0\rangle \qquad (40)$$

is given as

$$
\begin{aligned}
\langle A_{\mu\nu\delta}|B\rangle \;=\;& \mathcal{N}\int d^2a d^2b\delta(|a^2|-|b^2|-1)\exp\left(-\frac{\alpha'}{2}x^\mu U_{\mu\nu}x^\nu\right) \\
& \exp\left(-\sum_{n=1}^{\infty}\frac{\alpha'}{2}p_\mu M_{-n0}\frac{1}{n}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{n}}\right)_{\mu\nu}M^*_{-n0}p_\nu\right) \\
& A_{\mu\nu\delta}\sqrt{\frac{\alpha'}{2}}\left[\sum_n abM_{-n-a}\Lambda^{\mu\nu}(n)M^*_{-n-b}\sum_m cM_{-m0}\Lambda^{\alpha\delta}(m)M^*_{-m-c}p_\alpha\right. \\
& +\sum_n acM_{-n-a}\Lambda^{\mu\delta}(n)M^*_{-n-c}\sum_m bM_{-m0}\Lambda^{\alpha\nu}(m)M^*_{-m-b}p_\alpha \\
& -\frac{\alpha'}{2}\sum_n aM_{-n-a}\Lambda^{\mu\alpha}(n)M^*_{-n0}\sum_m bM_{-m0}\Lambda^{\beta\nu}(m)M^*_{-m-b} \\
& \left.\times\sum_l cM_{-l0}\Lambda^{\gamma\delta}(l)M^*_{-l-c}p_\alpha p_\beta p_\gamma\right], \qquad (41)
\end{aligned}
$$

and we have introduced the notational simplification

$$\frac{1}{m}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{m}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{m}}\right)^{\mu\nu} = \Lambda^{\mu\nu}(m). \qquad (42)$$

We would like to compare (16) to (41) in the same manner that we have for the tachyon and the massless states, and to do so requires a simple calculation. It can be shown that

$$\frac{1}{(k-1)!}\partial^k z^d(y)|_{y=0} = kM^*_{-d-k}. \tag{43}$$

This shows that the result obtained from differentiating the greens function, as deformed in the sigma model, gives the same result as the overlap of a closed string oscillator mode with the boundary state. Finally, again direct comparison of (16) to (41) gives $\mathcal{N} = Z$ again. This shows that our boundary state $|B\rangle$ gives the correct overlap with any closed string state, either on shell or off shell, and has fixed the normalization to be equal to the partition function.

4 One loop Boundary States

Now that we have fixed the normalization of the boundary states, and provided a consistent prescription for the action of the arbitrary PSL(2,R) transformation, we turn to the more intricate subject of the calculation of the overlap of two such states. This calculation can be thought of as a tree level exchange of closed strings between two D-branes with arbitrary field content. The naive expectation is the following, that the resulting expression will be the contribution of all the possible on and off shell one particle states that the boundary state can emit, weighted by the closed string propagator. It is interesting to note that, depending on U, not all possible physical closed string excitations are produced by the boundary state, and that in general states that do not satisfy the physical state condition are created.

Explicitly the thing we wish to calculate is the open string one loop correction to the partition function. The disk level correction was given in (8) and the one loop correction is given in the sigma model calculation by using the propagator on an annulus world-sheet, however in the boundary state representation the calculation is

$$\begin{aligned} Z_{One\ loop} &= \int d^2ad^2b\delta(|a^2|-|b^2|-1)d^2a'd^2b'\delta(|a^{'2}|-|b^{'2}|-1)\times \\ &\quad \mathcal{N}^2\langle B_{a',b'}|\frac{1}{L_0+\tilde{L}_0-2}|B_{a,b}\rangle. \end{aligned} \tag{44}$$

In the above, $|B_{a,b}\rangle$ is as given in (33). Note that this formulation will explicitly give factorization of the amplitude in the closed string channel.

The calculation of $Z_{One\ loop}$ is a straightforward, albeit tedious exercise. Using an integral representation for the propagator we find

$$\begin{aligned} Z_{One\ loop} &= \int d^2ad^2b\delta(|a^2|-|b^2|-1)d^2a'd^2b'\delta(|a^{'2}|-|b^{'2}|-1)\int_0^\infty dt\,\times \\ &\quad \mathcal{N}^2\langle B_{a',b'}|e^{-t(L_0+\tilde{L}_0-2)}|B_{a,b}\rangle. \end{aligned} \tag{45}$$

Now, suppressing for the moment the integrals, it is necessary to calculate the inner product itself, that is

$$\begin{aligned}\langle B_{a',b'}|e^{-t(L_0+\tilde{L}_0-2)}|B_{a,b}\rangle &= \langle -i\frac{\alpha'}{2}U^{\mu\nu}x_\nu|\exp\left(\sum_{n=1,j,k=-\infty}^{\infty}\alpha_k^\mu M_{-n-k}^{(1)*}\Lambda_{\mu\nu}(n)M_{-nj}^{(1)}\tilde{\alpha}_j^\nu\right)\\ &\quad \exp\left(-t\sum_{n\geq 1}(\alpha_{-n}^\mu\alpha_{n\mu}+\tilde{\alpha}_{-n}^\mu\tilde{\alpha}_{n\mu})-\frac{t\alpha'}{4}p^\mu p_\mu\right)\\ &\quad \exp\left(\sum_{n=1,j,k=-\infty}^{\infty}\alpha_{-k}^\mu M_{-n-k}^{(2)}\Lambda_{\mu\nu}(n)M_{-nj}^{(2)*}\tilde{\alpha}_{-j}^\nu\right)|-i\frac{\alpha'}{2}U^{\mu\nu}x_\nu\rangle.\end{aligned} \tag{46}$$

In this, we have denoted the dependence on a particular SL(2,R) transform by a superscript on the appropriate matrix, and abbreviated the Gaussian term in x acting on the Fock space vacuum. The above expression can be simplified considerably by using the relation

$$e^A e^B = e^B\left(\prod_{n=1}^{\infty}e^{\frac{1}{n!}[A,\ldots,[A,B]]}\right)e^A \tag{47}$$

which holds when $[A,B]$, $[A,[A,B]]$ and all similarly nested commutators commute with each other and with B, but do not commute with A. This then gives

$$\begin{aligned}\langle B_{a',b'}|e^{-t(L_0+\tilde{L}_0-2)}|B_{a,b}\rangle &= \langle -i\frac{\alpha'}{2}U^{\mu\nu}x_\nu|\exp\left(\sum_{n=1,j,k=-\infty}^{\infty}\alpha_k^\mu M_{-n-k}^{(1)*}\Lambda_{\mu\nu}(n)M_{-nj}^{(1)}\tilde{\alpha}_j^\nu\right)\\ &\quad \exp\left(-\frac{t\alpha'}{4}p^\mu p_\mu\right)\exp\left(\sum_{n=1,j,k=-\infty}^{\infty}\alpha_{-k}^\mu e^{-tk}M_{-n-k}^{(2)}\Lambda_{\mu\nu}(n)\right.\\ &\quad \left. M_{-nj}^{(2)*}\tilde{\alpha}_{-j}^\nu e^{-tj}\right)|-i\frac{\alpha'}{2}U^{\mu\nu}x_\nu\rangle.\end{aligned} \tag{48}$$

From here it is a straightforward exercise in combinatorics and commutation, very reminiscent of proofs of Wick's theorem, to obtain

$$\begin{aligned}\langle B_{a',b'}|e^{-t(L_0+\tilde{L}_0-2)}|B_{a,b}\rangle &= e^{2t}\exp\sum_k\frac{1}{k}\delta_\mu^\nu\delta_{mn}\left(\left[M_{-m-a}^{(2)}ae^{-ta}M_{-k-a}^{(1)*}\Lambda_\alpha^\mu(k)\right.\right.\\ &\quad \left.\left. M_{-k-b}^{(1)}be^{-tb}M_{-n-b}^{(2)*}\Lambda_\nu^\alpha(n)\right]^k\right)_{mn}^{\mu\nu}F(x)\end{aligned} \tag{49}$$

with matrix multiplication implied within the sum for both the Lorentz and oscillator indices, and $F(x)$ a Gaussian that depends on x, U, F and the transformation parameters. It can be verified that

$$
\begin{aligned}
F(x) \; = \; & \exp p^\mu p^\nu \left(-\frac{t\alpha'}{4} g_{\mu\nu} + M^{(1)*}_{-k0} \Lambda^\mu_\alpha(k) M^{(1)}_{-k-b} \times \right. \\
& \frac{1}{1 - be^{-tb} M^{(2)*}_{-n-b} \Lambda^\alpha_\beta(n) M^{(2)}_{-n-a} ae^{-ta} M^{(1)*}_{-p-a} \Lambda^\beta_\gamma(p) M^{(1)}_{-p-q}} qe^{-tq} M^{(2)*}_{-r-q} \Lambda^\gamma_\nu(q) M^{(2)}_{-q0} + \\
& M^{(1)*}_{-k0} \Lambda^\mu_\alpha(k) M^{(1)}_{-k-b} \frac{1}{1 - be^{-tb} M^{(2)*}_{-n-b} \Lambda^\alpha_\beta(n) M^{(2)}_{-n-a} ae^{-ta} M^{(1)*}_{-p-a} \Lambda^\beta_\nu(p) M^{(1)}_{-p-0}} + \\
& \left. \frac{1}{1 - M^{(2)*}_{-n0} \Lambda^\mu_\beta(n) M^{(2)}_{-n-a} ae^{-ta} M^{(1)*}_{-p-a} \Lambda^\beta_\gamma(p) M^{(1)}_{-p-q}} qe^{-tq} M^{(2)*}_{-r-q} \Lambda^\gamma_\nu(q) M^{(2)}_{-q0} \right), \quad (50)
\end{aligned}
$$

with $p^\mu = -\frac{i\alpha'}{2} U^\mu_\nu x^\nu$ as in 21 . It should be noted that this seemingly complicated expression is nothing but a Gaussian in x, and so just has the effect of localizing the interaction between two D-branes. The final expression for $Z_{One\ loop}$ is obtained by integrating (49) over the elements of the conformal transformations noted above.

It is instructive to examine the form of this. First, note that ignoring the x dependence we can express as

$$
\begin{aligned}
Z_{One\ loop} \; = \; & \int d^2a d^2b \delta(|a^2| - |b^2| - 1) d^2a' d^2b' \delta(|a'^2| - |b'^2| - 1) \times \\
& \mathcal{N}^2 e^{2t} \frac{1}{\det \left(1 - be^{-tb} M^{(2)*}_{-n-b} \Lambda^\alpha_\beta(n) M^{(2)}_{-n-a} ae^{-ta} M^{(1)*}_{-p-a} \Lambda^\beta_\gamma(p) M^{(1)}_{-p-q} \right)}
\end{aligned}
\quad (51)
$$

In the cases of $U \to 0$ and $U \to \infty$ the matrices $M^*\Lambda M$ revert to a particularly simple form. We have

$$
\begin{aligned}
\left. \sum_{n\geq 1} M^*_{-n-b} \Lambda^{\mu\nu}(n) M_{-n-a} \right|_{(U\to 0)} &= \sum_{n\geq 1} M^*_{-n-b} \frac{1}{n} \left(\frac{g - 2\pi\alpha' F}{g + 2\pi\alpha' F} \right)^{\mu\nu} M_{-n-a} \\
&= \left(\frac{g - 2\pi\alpha' F}{g + 2\pi\alpha' F} \right)^{\mu\nu} \frac{1}{b} \delta_{ab}
\end{aligned}
\quad (52)
$$

and also

$$
\left. \sum_{n\geq 1} M^*_{-n-b} \Lambda^{\mu\nu}(n) M_{-n-a} \right|_{(U\to\infty)} = -g^{\mu\nu} \frac{1}{b} \delta_{ab} \quad (53)
$$

which follow from the definition of M. We can see that in the case of $U = 0$ that the boundary state for a background gauge field is recovered, and in the case $U = \infty$ the boundary state for a localized object, a D-brane, is recovered. In both the cases the integral over the volume of PSL(2,R) becomes a trivial prefactor, as expected. It is also possible to say something about the more general case. The sum $M^{(2)}_{-n-a} a e^{-ta} M^{(1)*}_{-q-a}$ can be taken over a and we find the result is a transformation in SL(2,C), of which PSL(2,R) is a subgroup. Explicitly

$$\sum_{a\geq 1} M^{2}_{-n-a} a e^{-ta} M^{(1)*}_{-q-a} = \oint \frac{dz}{2\pi i}\frac{1}{z^n}\frac{\left[z(a_1^* e^{-t/2}a_2 - b_1 e^{t/2} b_2^*) + (a_1^* e^{-t/2} b_2 - b_1 e^{t/2} a_2^*)\right]^{q-1}}{\left[z(a_1 e^{t/2} b_2^* - b_1^* e^{-t/2} a_2) + (a_1 e^{t/2} a_2^* - b_1^* e^{-t/2} b_2)\right]^{q+1}}. \tag{54}$$

The fact that the integral over conformal factors becomes an integral over a transformation in a larger group is appropriate. The integral over t stands in the place of the integral over the Teichmuller parameter of the annulus, and the other degrees of freedom correspond to reparametrizations of the two ends of the annulus.

It is also interesting to examine how these expressions for $Z_{One\ loop}$ vary with U around the two fixed points. In particular, ignoring the linear terms in U in the normalization, which can be seen (8) to be divergent, the expression for $Z_{One\ loop}$ near $U=0$ is

$$Z_{One\ loop} = Z_{One\ loop}(U=0) + Tr\left(U\frac{\partial}{\partial U} Z_{One\ loop}(U=0)\right) + \ldots \tag{55}$$

Immediately upon differentiation we see that the linear term will be given by

$$\begin{aligned} Tr\left(U\frac{\partial}{\partial U} Z_{One\ loop}(U=0)\right) &= \int d^2a d^2b \delta(|a^2| - |b^2| - 1)\mathcal{V}_{PSL(2,R)}\mathcal{N}^2 e^{2t} \\ &\quad \frac{1}{\det\left(1 - e^{-2tb}\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)^2\right)} \\ &\quad \times Tr\left(\frac{(g-2\pi\alpha' F)/(g+2\pi\alpha' F)}{(g+2\pi\alpha' F)^2 - (g-2\pi\alpha' F)^2} U\right) \\ &\quad \times \frac{-4be^{-2b}}{1-e^{-2b}} M^*_{-n-b}\frac{1}{n^2} M_{-n-b}. \end{aligned} \tag{56}$$

The factor of $\frac{e^{-2b}}{1-e^{-2b}}$ comes from the fact that all the other Λ terms become trivial because we have evaluated them at $U=0$ which was noted to be conformally invariant, and from summing the terms e^{-b} which stand between these. Likewise note that the factor $1/n^2$ instead of $1/n$ between M^* and M comes from the fact that U enters always as U/n. Also, one of the integrals over the PSL(2,R) groups becomes trivial, and relabeling gives the factor $\mathcal{V}_{PSL(2,R)}$ and only one integral. Now, to calculate explicitly

$$\sum_{n\geq 1} M^*_{-n-b}\frac{1}{n^2} M_{-n-b} = \sum_{n\geq 1}\oint \frac{dz}{2\pi i}\frac{d\bar{z}}{-2\pi i}\frac{1}{n^2}\frac{1}{z^n \bar{z}^n}$$

$$\frac{(a^*\bar{z}+b^*)^{b-1}}{(b\bar{z}+a)^{b+1}}\frac{(az+b)^{b-1}}{(b^*z+a^*)^{b+1}}$$

$$\frac{1}{n!^2}\partial_z^{n-1}\partial_{\bar{z}}^{n-1}\frac{(a^*\bar{z}+b^*)^{b-1}}{(b\bar{z}+a)^{b+1}}\frac{(az+b)^{b-1}}{(b^*z+a^*)^{b+1}}\Bigg|_{z,\bar{z}=0} \tag{57}$$

This can be calculated explicitly, and when we include the integration over PSL(2,R) we find that it becomes

$$\begin{aligned}
&\int d^2ad^2b\delta(|a^2|-|b^2|-1) \\
&\times\sum_{n\geq 1}M^*_{-n-b}\frac{1}{n^2}M_{-n-b} = \int d^2ad^2b\delta(|a^2|-|b^2|-1)\sum_{n\geq 1}\sum_{q=0}^{min(n-1,b-1)}\frac{1}{(nb)^2} \\
&\qquad\left(\frac{b+n-q-1!}{q!n-q-1!b-q-1!}\right)^2\left(\frac{|b^2|}{|a^2|}\right)^{b+n-2q-2}\frac{1}{|a^2|^2}
\end{aligned} \tag{58}$$

and we have used the fact that upon integration over the phase of a and b we will have orthogonality in the sum. All in all the contribution is

$$\begin{aligned}
Tr\left(U\frac{\partial}{\partial U}Z_{One\ loop}\right)_{U=0} &= \int d^2ad^2b\delta(|a^2|-|b^2|-1)\mathcal{V}_{PSL(2,R)}\mathcal{N}^2e^{2t} \\
&\times\frac{1}{\det\left(1-e^{-2tb}\left(\frac{g-2\pi\alpha'F}{g+2\pi\alpha'F}\right)^2\right)} \\
&\times Tr\left(\frac{(g-2\pi\alpha'F)/(g+2\pi\alpha'F)}{(g+2\pi\alpha'F)^2-(g-2\pi\alpha'F)^2}U\right)\sum_{n,b\geq 1}\frac{-4be^{-2b}}{1-e^{-2b}} \\
&\times\sum_{q=0}^{min(n-1,b-1)}\frac{1}{(nb)^2}\left(\frac{b+n-q-1!}{q!n-q-1!b-q-1!}\right)^2 \\
&\left(\frac{|b^2|}{|a^2|}\right)^{b+n-2q-2}\frac{1}{|a^2|^2}.
\end{aligned} \tag{59}$$

A similar calculation can be done around the d-brane ($U\to\infty$) and we find that

$$\begin{aligned}
Tr\left(\frac{1}{U}\frac{\partial}{\partial(1/U)}Z_{One\ loop}\right)_{\frac{1}{U}=0} &= \int d^2ad^2b\delta(|a^2|-|b^2|-1)\mathcal{V}_{PSL(2,R)}\frac{\mathcal{N}^2e^{2t}}{\det\left(1-e^{-2tb}\right)} \\
&\times 4Tr\left(\frac{1}{U}\right)\sum_{n,b\geq 1}\frac{be^{-2b}}{1-e^{-2b}}M^*_{-n-b}M_{-n-b}.
\end{aligned} \tag{60}$$

Because the natural coefficient for $\frac{1}{U}$ is n the n dependence between the matrices M is suppressed. Explicit evaluations show that $M^*_{-n-a}M_{-n-b}$ has zero entries on diagonal, so this variation vanishes about the d-brane. This comparison between (59) and (60) shows that the case of Neumann boundary conditions, (corresponding to $U \to 0$) is unstable with respect to variations of the tachyon condensate since the linear variation does not vanish, but that Dirichlet boundary conditions, obtained as $U \to \infty$ are stable. This illustrates the well known phenomenon of tachyon condensation and gives a mechanism to see explicitly how the open string tachyon has been removed from the excitations of the condensed state.

5 Conclusions

In this note we have presented a generalization of the boundary state formalism that allows us to calculate the overlap of the boundary state with arbitrary on and off shell closed string states. We have shown that it exactly reproduces the calculations that would be done in a sigma model for an appropriate vertex operator, verifying the conjecture of [**?**]. This generalization gives a prescription for the calculation of the boundary state annulus amplitude which reproduces the expected modular transformation structure and explicitly factorizes in the closed string channel.

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